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On the flow of an elastico-viscous fluid near a rotating disk

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Abstract

The steady laminar flow of an elastico-viscous fluid near a rotating disk is considered. The constitutive equations of the fluid are modeled by those for a Walter B′ fluid. They give rise to a boundary-value problem in which the order of the system of differential equations is seven but there are only five boundary conditions. Nevertheless, without augmenting any boundary condition at infinity, it is possible to obtain an exact numerical solution for any value of k, the viscoelastic fluid parameter. The solution also takes into account and eliminates the error introduced by replacing numerical infinity with a finite value.

It is shown that solutions exist for all values of k. A perturbation solution valid for small values of k, and an asymptotic analytical solution valid for large values of k are also derived, each up to the third order. By comparing them with the exact solution, a critical assessment is undertaken of their respective domains of usefulness.

Keywords: Elastico-viscous fluids; Walters B′ fluid; Rotating disk; Numerical solution; Boundary value problem

1. Introduction

The flows of viscoelastic fluids present some interesting challenges to researchers in engineering, applied mathematics and computer science. The constitutive equations of viscoelastic fluids can be very complex involving a number of parameters, but even the simplest of these, for example, those based on Walter B′ model can give rise to problems which are far from trivial. In the flows in which there is a cross-flow velocity component, typically, they lead to boundary-value problems (BVPs) in which the order of differential equations exceeds the number of available boundary conditions. Ever since the simplest models of viscoelastic fluids were introduced, the researchers in rheology have been looking for the elusive boundary condition(s) associated with the viscoelastic fluid parameter, but essentially without success. Accordingly, research during the last few decades has centered on

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developing algorithms which are able to deal with the supposedly ill-posed problems of the type mentioned above.

In the initial epoch of research in flows of viscoelastic fluids almost all efforts were directed at getting the solution of the BVPs using the perturbation technique. Beard and Walters [8] in their basic problem of stagnation point flow of an elastico-viscous fluid indeed used the perturbation analysis and arrived at the startling conclusion that the velocity in the boundary-layer flow exceeds that in the potential flow. Serth [17] was probably the first to obtain the solution of the BVP considered in [8] without placing any restriction on the size of k, the viscoelastic fluid parameter. He observed a rather disconcerting trend of stress at the wall rising sharply as the value of k was increased. Teipel [19], using a Taylor series expansion near $\eta=0$ (η being the distance from the wall) and the Runge–Kutta method beyond a suitably chosen 'patch' point was able to give an accurate numerical solution of the problem of Beard and Walters [8]. He showed (i) that the solution exhibited an oscillatory behavior for large η, causing the flow to overshoot the mainstream velocity, and (ii) that the solution could be obtained only up to certain critical value of k, say k_c. Teipel [18] also analyzed the axisymmetric stagnation point flow of a second-order fluid [15] which has a slightly more generalized constitutive equation, and arrived at similar conclusions. Teipel, in trying to explain the nonexistence of solutions for $k > k_c$, surmised that at $k = k_c$ the stresses tend to become unbounded and he backed up his arguments by deriving an approximate solution for the two-dimensional (2D) stagnation point flow based on the Von Kármán–Polhousen integral approach, which, as is well known, gives highly satisfactory results for Newtonian fluids. There is yet another aspect of Teipel's investigation that deserves to be mentioned. Though Teipel did not report any problems with integration, there is a reason to believe that there can be numerical instability for small values of k (and η). This is on account of the fact that the coefficient of the highest derivative in the equation of motion is $O(k\eta^2)$ for small values of η, and this can cause severe problems if k is also very small, say, $O(10^{-3})$. The lowest nonzero value of k reported by Teipel was 0.1; therefore, it is likely that he might not have encountered the problem of numerical instability.

The present author [2] has suggested an algorithm for computing the flow of viscoelastic fluids which obviates the above-mentioned difficulties. The algorithm is applicable for *all* values of k for which the solution exists, and this includes the case $k = 0$, and the case of vanishingly small values of k. In his paper he was able to demonstrate the viability of the algorithm by computing three important flows: (i) 2D stagnation point flow, (ii) 2D flow due to a stretching sheet, and (iii) flow due to a rotating disk. For the first flow it was discovered that the solution has a turning point, implying that dual solutions exist up to $k = k_c$ and that no solution exists beyond $k = k_c$. Similar results were obtained by the author for the problems of axisymmetric stagnation point flow for an elasticoviscous fluid [3] and the laminar flow of a viscoelastic fluid through a channel in which one wall is porous and the other wall is impermeable [4]. For the second flow discussed in [2], an exact solution exists and it provides a benchmark for the algorithm. The results produced using the algorithm were in complete agreement with those given by the exact solution. The results for the third flow—the flow due to a rotating disk—showed that an earlier analysis by Elliott [10], based on perturbation analysis failed to predict correctly the behavior of the turning moment (or torque) on the disk with k. Whereas the perturbation analysis suggests that the turning moment increases with k, the actual numerical results suggest that this conclusion is valid only for very small values of k and that for larger values of k, the oppositive conclusion holds.

The main emphasis in [2] was on the development of an algorithm that would be able to handle successfully the inherent problems associated with the integration of BVPs governing the flow of the viscoelastic fluids, namely, having the number of available boundary conditions less than the order of differential equations, and the highest derivative being multiplied by the viscoelastic fluid parameter. Therefore, several other issues relating to the flow were not considered in detail. This is especially true in relation to the flow of an elastico-viscous fluid near a rotating disk. Some of these issues are: (i) Does the solution exist for all values of k? (ii) If the solution exists for all values of k, does it exhibit an asymptotic behavior for large k? (iii) Since the torque on the disk first increases and then decreases and there does not seem to be a turning point in the solution, will a higher-order perturbation solution adequately represent this phenomenon? (iv) Is it possible to negotiate with the problem associated with ζ_∞, the numerical infinity? It is well known that replacing infinity with a finite value ζ_∞ leads to an error in the solution. Several techniques have been suggested in the literature to minimize or even eliminate this source of error.

Keeping the above-mentioned issues in mind we reexamine the laminar flow of an elastico-viscous fluid near a rotating disk. Firstly, we develop an algorithm which takes into account the need of eliminating the replacement of infinity by a finite value. Secondly, we present higher perturbation solutions valid for small values of k up to the order of k^3. Finally, having concluded that the solutions exist for all values of k, we present the analytical asymptotic solutions valid for large values of k. This incidentally also paves the way for obtaining an efficient iterative numerical solution for large values of k, as some difficulties were encountered in obtaining the solution using the main algorithm for values of k beyond some critical value.

The companion problem for the flow of a second-grade fluid near a rotating disk has been recently investigated by the author [6]. For earlier references on the flow near a rotating disk under varying physical settings, one may refer to the citations listed in [6].

2. Equations of motion

The constitutive equations of the Walter B′ elastico-viscous fluid are

$$p_{ik} = -pg_{ik} + \tau_{ik}, \tag{1}$$

where p_{ik} is stress tensor, p is isotropic pressure, g_{ik} is the metric tensor of a fixed coordinate system x^i, and for fluids with short memories (i.e., short relaxation times), τ_{ik} is defined by

$$\tau^{ik} = 2\eta_0 d^{ik} - 2k_0 \bar{\tau}^{ik}, \tag{2}$$

d^{ik} being the rate of strain tensor, given by

$$d^{ik} = \tfrac{1}{2}(v^i_{,k} + v^k_{,i}), \tag{3}$$

and $\bar{\tau}^{ik}$ is given by

$$\bar{\tau}^{ik} = \frac{\partial d^{ik}}{\partial t} + d^{ik}_{,j} v^j - d^{ij} v^k_{,j} - d^{jk} v^i_{,j} + d^{ik} v^k_{,k}. \tag{4}$$

Lastly $\eta_0 = \int_0^\infty N(\tau)\,\mathrm{d}\tau$ is the limiting viscosity at small rates of shear, and $k_0 = \int_0^\infty \tau N(\tau)\,\mathrm{d}\tau$ is the short-memory coefficient. Here $N(\tau)$ is the distribution function of relaxation time τ. In the derivation of Eq. (2) the terms involving $\int_0^\infty \tau^n N(\tau)\,\mathrm{d}\tau$, $n \geqslant 2$ have been neglected.

With the stress tensor defined by (1), equations of motion can be written as

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\eta_0\nabla^2\mathbf{v}-k_0\left(\frac{\partial}{\partial t}\nabla^2\mathbf{v}+2(\mathbf{v}\cdot\nabla)\nabla^2\mathbf{v}-\nabla^2(\mathbf{v}\cdot\nabla)\mathbf{v}\right), \tag{5}$$

$$\nabla\cdot\mathbf{v}=0, \tag{6}$$

where ρ and $\mathbf{v}$ are, respectively, the density and velocity vector of the fluid at a point.

We consider the laminar flow of a Walter B$'$ fluid near a rotating disk which is assumed to be rotating in the plane $z=0$ about z-axis with a uniform velocity ω. The flow takes place in the upper half plane $z>0$. For a Newtonian fluid Van Kármán [20] was the first to demonstrate that the transformations

$$u=r\omega F(\zeta),\quad v=r\omega G(\zeta),\quad w=\sqrt{\frac{\eta_0\omega}{\rho}}H(\zeta), \tag{7}$$

where (u,v,w) are the components of the velocity vector $\mathbf{v}$ in the cylindrical polar coordinate system (r,θ,z) with origin at the center of the disk, and

$$\zeta=\sqrt{\frac{\rho\omega}{\eta_0}}z, \tag{8}$$

reduce the Navier–Stokes equations for a Newtonian fluids to a set of ordinary differential equations in F, G and H. Same is the case with the Walter B$'$ fluid, and Elliott [10] was able to obtain the following BVP:

$$F''-HF'-F^2+G^2-k(HF'''+4FF''+2G'^2)=0, \tag{9}$$

$$G''-HG'-2FG-k(HG'''+4FG''-2F'G')=0, \tag{10}$$

$$2F+H'=0, \tag{11}$$

with boundary conditions

$$F(0)=0,\quad G(0)=1,\quad H(0)=0, \tag{12}$$

$$F(\infty)=0,\quad G(\infty)=0. \tag{13}$$

We now present an exact numerical solution of system (9)–(13) without placing any restriction on the value of k.

3. An exact numerical solution

Note that the system of equations (9)–(11) has the order seven, but there are only five boundary conditions in (12) and (13). It is indeed possible to supplement two extra conditions

$$F'(\infty)=0,\quad G'(\infty)=0 \tag{14}$$

and arrive at a 'consistent' set of equations in the BVP. However, it would be unthematic. For example, if one considers the flow between one rotating and one fixed disk, no such condition as

(14) can be imposed on the fixed disk. Hence, if possible, we should try to obtain a solution without taking recourse to boundary conditions such as those given in (14).

The present author [2] has given an effective algorithm for handling such problems. We shall be following the same methodology, in principle, at any rate. However, there is a very significant factor that must be taken into account. When the results were computed in [2] for a rotating disk, it was discovered that it took longer distances for F, G and H to approach their asymptotic values, as the value of k was increased. This implies that the value of ζ_∞, the numerical infinity, must be increased as k is increased. Now for a Newtonian fluid there is already an error due to the replacement of infinity by a finite value of ζ_∞. Evidently, this error is going to increase further unless the value of ζ_∞ is properly adjusted with a change in the value of k.

There are quite a few ways of handling the problem mentioned above. One of them is the free boundary-value problem (FBVP) formulation in which the higher-order derivatives at ζ_∞ are assigned some arbitrarily small values, and ζ_∞ is calculated in the solution process. As a bonus, one is able to obtain the values of the missing initial conditions to almost machine precision. Fazio [11] and Ariel [5] derived the FBVP formulation of two important flow problems, namely, Blasius flow and the stagnation point flow, respectively, and negotiated successfully the errors caused by the replacement of infinity by a finite numerical value.

While the FBVP formulation could be the appropriate technique for handling the boundary-layer flows in infinite domains, for flows due to moving boundaries, there is an excellent alternative in the form of series expansion in terms of exponentially decaying functions. Cochran [9] suggested the said expansion for the flow of a Newtonian fluid near a rotating disk. However, he restricted the application of the expansion only to large values of ζ. For small values of ζ he developed another expansion and matched the solution at some intermediate value of ζ, where both expansions were valid. This technique, though it obviates the necessity of replacement of infinity by a finite value, is highly cumbersome, as it involves solving iteratively for five parameters. Ackroyd [1] ventured to extend the solution involving exponentials all the way to the disk, and found that it was possible to match the boundary conditions there. This had the important consequence of completely eliminating the error caused by the replacement of infinity by a finite value. In the following we shall be following Ackroyd's idea and thereby refine the algorithm given in [2].

The key factor in Ackroyd's method is the presence of a scale factor c that characterizes the flow caused by moving boundaries. Thus, the solution can be expressed in terms of

$$\chi = \mathrm{e}^{-c\zeta}. \tag{15}$$

Writing

$$F = \sum_{n=1}^{\infty} a_n \chi^n, \quad G = \sum_{n=1}^{\infty} b_n \chi^n, \tag{16}$$

which in conjunction with Eq. (11) implies

$$H = H_\infty + \frac{2}{c} \sum_{n=1}^{\infty} \frac{a_n}{n} \chi^n, \tag{17}$$

(H_∞ being the axial velocity at infinity), and substituting in Eqs. (9) and (10), we obtain

$$c + H_\infty + kc^2 H_\infty = 0 \tag{18}$$

and

$$(c^2n^2 + cnH_\infty + kc^3n^3H_\infty)a_n + \sum_{m=1}^{n-1}\left\{\left[\left(\frac{2n}{m} - 3\right)a_m a_{n-m} + b_m b_{n-m}\right]\right.$$
$$\left. + 2kc^2(n-m)\left[\left(\frac{n^2}{m} - 4n + 3m\right)a_m a_{n-m} - m b_m b_{n-m}\right]\right\} = 0, \tag{19}$$

$$(c^2n^2 + cnH_\infty + kc^3n^3H_\infty)b_n + \sum_{m=1}^{n-1} 2(n-2m)\left[\frac{1}{m} + (n-m)\left(\frac{n}{m} - 2\right)kc^2\right]a_m b_{n-m} = 0. \tag{20}$$

For a Newtonian fluid ($k = 0$) Eqs. (18)–(20) reduce to the corresponding equations obtained by Ackroyd [1]. From our point of view, Eq. (18) is the most important as it connects the scale factor c to H_∞, the axial velocity at infinity. Solving Eq. (18) for c, we obtain

$$c = -\frac{2H_\infty}{1 + \sqrt{1 - 4kH_\infty^2}}. \tag{21}$$

In obtaining (21), the proper sign has been chosen in the solution of the quadratic equation (18), so that corresponding to $k = 0$, the value of c matches with $-H_\infty$, the solution for the Newtonian fluids.

As pointed out by the author [6] for the companion problem of flow of a second-grade fluid near a rotating disk, it is highly doubtful if the series for F and G defined by Eqs. (16) will converge for $k \neq 0$, except perhaps for a very restricted range of ζ greater than some critical value, that depends on k. We need not resort to the series solution; however, we can simply use relation (15) for mapping the infinite domain of integration $[0, \infty)$ for ζ to the fine domain $[0, 1]$ for χ and then integrate the resulting equations numerically with respect to χ.

If we define

$$F' = S \quad \text{and} \quad G' = T \tag{22}$$

in accordance with the spirit of algorithm given in [2], the following system is obtained for S and T:

$$S' - HS - F^2 + G^2 - k(HS'' + 4FS' + 2T^2) = 0, \tag{23}$$

$$T' - HT - 2FG - k(HT'' + 4FT' - 2ST) = 0. \tag{24}$$

Now we use transformation (15) to write the corresponding system in the domain $0 \leqslant \chi \leqslant 1$. We get

$$-c\chi\frac{dS}{d\chi} - HS - F^2 + G^2 - k\left[c^2H\chi\frac{d}{d\chi}\left(\chi\frac{dS}{d\chi}\right) - 4cF\chi\frac{dS}{d\chi} + 2T^2\right] = 0, \tag{25}$$

$$-c\chi\frac{dT}{d\chi} - HT - 2FG - k\left[c^2H\chi\frac{d}{d\chi}\left(\chi\frac{dT}{d\chi}\right) - 4cF\chi\frac{dT}{d\chi} - 2ST\right] = 0, \tag{26}$$

$$-c\chi\frac{dF}{d\chi} = S, \tag{27}$$

$$-c\chi\frac{dG}{d\chi}=T, \tag{28}$$

$$2F-c\chi\frac{dH}{d\chi}=0. \tag{29}$$

Note that the introduction of the independent variable χ produces a singularity at $\chi=0$. De Hoog and Weiss [12,13] have suggested various difference schemes to solve the BVPs with singularities of both types—first kind and essential. One of the schemes is to use the trapezoidal rule. We have also used trapezoidal rule for Eqs. (27)–(29). However, for Eqs. (25) and (26) we used the idea that was found to be highly successful in solving singular BVPs in infinite domains [12].

Setting up the mesh

$$\chi_i=ih \quad (i=1,2,\ldots,N), \tag{30}$$

where h is the mesh size and N is an integer satisfying $Nh=1$, we obtain, on discretization of system (25)–(29)

$$\begin{aligned}&-\tfrac{1}{2}ic(S_{i+1}-S_{i-1})-H_iS_i-F_i^2+G_i^2-k\{ic^2H_i[(i+\tfrac{1}{2})(S_{i+1}-S_i)\\&\quad-(i-\tfrac{1}{2})(S_i-S_{i-1})]-2icF_i(S_{i+1}-S_{i-1})+2T_i^2\}=0,\end{aligned} \tag{31}$$

$$\begin{aligned}&-\tfrac{1}{2}ic(T_{i+1}-T_{i-1})-H_iT_i-2F_iG_i-k\{ic^2H_i[(i+\tfrac{1}{2})(T_{i+1}-T_i)\\&\quad-(i-\tfrac{1}{2})(T_i-T_{i-1})]-2icF_i(T_{i+1}-T_{i-1})-2S_iT_i\}=0,\end{aligned} \tag{32}$$

$$-c(i-1)(F_i-F_{i-1})=\tfrac{1}{2}(S_i+S_{i-1}), \tag{33}$$

$$-c(i-1)(G_i-G_{i-1})=\tfrac{1}{2}(T_i+T_{i-1}), \tag{34}$$

$$F_i+F_{i+1}-c(i-\tfrac{1}{2})(H_i-H_{i-1})=0, \tag{35}$$

with boundary conditions

$$F_0=0, \quad G_0=0, \quad H_0=H_\infty, \tag{36}$$

$$F_N=0, \quad G_N=1, \quad H_N=0. \tag{37}$$

In Eqs. (25) and (26) we have used the central differences for the derivatives at the mesh point i, but in Eqs. (27)–(29) the same have been used at the mesh point $i-\frac{1}{2}$, with the values of the function being replaced by the corresponding averages at the mesh points i and $i-1$. This scheme ensures that an accuracy of $O(h^2)$ is preserved in discretization.

Eqs. (31) and (32) are both three-term recurrence relations, whereas Eqs. (33)–(35) are two-term recurrence relations. To start the recursion in Eqs. (31) and (32), two values each are required, but only one value each is required in Eqs. (33)–(35). Thus, besides c, which is expressible in terms of H_∞ (see Eq. (21)), we need seven starting values to solve system (31)–(35). However, only six values are available from the boundary conditions (36) and (37) to determine these eight values. This is a typical characteristic of the flow problems of viscoelastic fluids.

The theme of not decomposing Eqs. (9) and (10) all the way down to the first-order system of equations has amongst others a highly desirable feature, namely, that the pair of equations (31) and (32) can be solved explicitly for either S_{i+1} (and T_{i+1}) or S_{i-1} (and T_{i-1}), as both the equations are linear in terms of the variable for which they are to be solved. Which of the two choices must be made depends on the coefficient of S_{i+1} or S_{i-1}—the important consideration being that the coefficient should *not* change its sign in the integration process, otherwise the latter would break down. For the second-grade fluid we found that the proper choice was to solve ‘forward’, i.e., for the values at the $(i+1)$th mesh point in terms of the values at the preceding two mesh points. Here, however, because of the opposite sign of the viscoelastic fluid parameter, we must solve ‘backward’. Thus, we have

$$S_{i-1} = \{\tfrac{1}{2}\,ic - k[ic^2(i-\tfrac{1}{2})H_i + 2icF_i]\}^{-1} \times (\{\tfrac{1}{2}\,ic + k[ic^2(i+\tfrac{1}{2})H_i - 2icF_i]\}S_{i+1} + (1-2ki^2c^2)H_iS_i + F_i^2 - G_i^2 + 2kT_i^2), \tag{38}$$

$$T_{i-1} = \left\{\tfrac{1}{2}\,ic - k\left[ic^2\left(i-\tfrac{1}{2}\right)H_i + 2icF_i\right]\right\}^{-1} \times \left(\left\{\tfrac{1}{2}\,ic + k\left[ic^2\left(i+\tfrac{1}{2}\right)H_i - 2icF_i\right]\right\}T_{i+1} + (1-2ki^2c^2)H_iT_i + 2F_iG_i - 2kS_iT_i\right), \tag{39}$$

$$F_{i-1} = F_i + \frac{S_i + S_{i-1}}{(2i-1)c}, \tag{40}$$

$$G_{i-1} = G_i + \frac{T_i + T_{i-1}}{(2i-1)c}, \tag{41}$$

$$H_{i-1} = H_i - \frac{2(F_i + F_{i-1})}{(2i-1)c}. \tag{42}$$

If we denote the missing initial conditions for the original BVP (9)–(13) as

$$\frac{\mathrm{d}F(0)}{\mathrm{d}\zeta} = s \quad \text{and} \quad \frac{\mathrm{d}G(0)}{\mathrm{d}\zeta} = t, \tag{43}$$

we have the following values for starting the recursion:

$$S_N = s, \quad T_N = t, \quad F_N = 0, \quad G_N = 0 \quad \text{and} \quad H_N = 0. \tag{44}$$

The values of s, t and c (or H_∞) must be so chosen that the terminal conditions (36) are satisfied.

As remarked earlier, for starting the recursion in Eqs. (38) and (39) we need two values each, thus we also need the values of S_{N-1} and T_{N-1}. These can be readily obtained by considering the Taylor series expansion of F and G near $\zeta = 0$. We have

$$F''(0) = -1 + 2k[G'(0)]^2, \quad G''(0) = -2kF'(0)G'(0),$$

$$F'''(0) = -2G'(0) + 4k[F'(0)F''(0) + G'(0)G''(0)],$$

$$G'''(0) = 2F'(0) + 2k[F'(0)G''(0) - F''(0)G'(0)],$$

$$\vdots$$

$$.$$

This gives

$$S_{N-1} = s - (1 - 2kt^2)\left(1 + \frac{1}{2}h\right)\frac{h}{c} - (t + 2ks)\frac{h^2}{c^2} \tag{45}$$

and

$$T_{N-1} = t - 2kst\left(1 + \frac{1}{2}h\right)\frac{h}{c} + \{s + kt[1 - 2k(s^2 + t^2)]\}\frac{h^2}{c^2}, \tag{46}$$

the error in each of the above being $O(h^3)$.

Thus, given s and t, one can obtain S_{N-1} and T_{N-1} from Eqs. (45) and (46). The stage is now set for obtaining the values of F, G, H, S and T at all levels. We assign values to s, t and c. This gives the values at level N. To obtain the values at level $N-1$ we compute the values of S and T from Eqs. (45) and (46), respectively, and those of F, G, and H, in that order, from Eqs. (40), (41) and (42), respectively. From this point on we can move to the next lower level repeatedly and compute the values of S, T, F, G and H, in that order, from Eqs. (38) to (42). The values of F, G and H so obtained at level '0' must match with those in Eq. (36). Thus, the problem reduces to that of zero-finding involving three parameters.

Note that even though we have used finite-difference schemes to approximate the derivatives we are still using a shooting method to solve the present BVP. In general, it is a well-established fact that the relaxation methods are more robust and better suited for solving BVPs which are sensitive to the missing initial conditions—and the present problem, as we shall see presently, *is* quite sensitive to the initial guesses for the missing conditions. There is a good coverage of the relaxation methods in the classical texts by Keller [14] and Ascher et al. [7]. It would be perhaps a good idea to solve the present problem using these methods, but this is not done for the reasons that the shooting method is straightforward and it works well for moderate values of k, which are usually encountered in practice, *and*, for larger values of k there is an alternate iterative scheme, which is equally attractive to implement.

Now if we were to follow the algorithm in [2], we would have to solve for two missing values of $F'(0)$ and $G'(0)$ only. Hence, we have to pay a price for effectively dealing with numerical infinity in terms of solving the problem for an extra parameter, namely, c or H_∞. On the positive side we have not only eliminated the need of replacing infinity with a finite value, but because of the transformation of the domain of integration from $(0,\infty)$ to $(0,1)$ we are in a position to obtain accurate solution without taking too many mesh points. The fact that the algorithm has an accuracy of only $O(h^2)$ need not concern us unduly as we can easily hike the accuracy to $O(h^4)$ by invoking Richardson's extrapolation. Another significant benefit is that there is no need to adjust the value of ζ_∞ which tends to increase substantially with k, a factor that would have to be taken into consideration if transformation (15) is not used.

For zero-finding several algorithms are available. We found the three-dimensional (3D) version of the secant method quite adequate, at least for zero to moderate values of k. It is true that compared to its 1D and 2D analogues, the 3D version requires a lot more iterations for the same accuracy. For the 2D version used in [2], 5–10 iterations were required for an accuracy of eight significant digits, the 3D version used in the present work required on an average 12–15 iterations. This in our opinion is a very reasonable premium to pay for the advantages outlined above that we get in return.

No problems were encountered in obtaining the solution for zero or very small values of k. However, as k was increased, the problem became increasingly sensitive to the starting values of s, t and H_∞. Note in particular the restriction imposed on H_∞ in Eq. (21), namely $|H_\infty| \leqslant 1/2\sqrt{k}$. For large values of k, it was not uncommon for H_∞ to stray in the forbidden domain, but then the value of c was simply chosen as $-2H_\infty$ by setting the value of the quantity under the square root sign to zero. It was a self-correcting process and eventually when the iterations did converge, the proper value of c (or H_∞) was obtained. With a step size of $h = 0.01$, we were able to obtain the solution with reasonable effort, for values of k up to 15, though for higher values of k in this range, we had to use the trial values that were very close to the exact values. The method of continuation was used, i.e., the value of k was incremented uniformly and the missing values of s, t and H_∞ were estimated using a polynomial extrapolation in k up to the degree four. The step size had to be reduced to 0.005 in order to obtain the solution for values of k between 15 and 20. Above 20, the algorithm became so sensitive to the starting values that even a difference of 10^{-8} in the values caused the iterations to diverge. The situation can perhaps be retrieved to some extent by using multiple precision and higher-order extrapolating polynomials. However, it is not worthwhile to pursue it, especially since there is an alternate effective algorithm for large values of k. Since it is intimately connected to the asymptotic solution for large k, we defer giving its details presently. Nevertheless, we emphasize that the two algorithms taken together allow us to find the solutions for all values of $k \geqslant 0$. We can, therefore, conclude that unlike the other flow problems of Walters 'B′ fluid reported in the literature, such as the stagnation point flow, flow due to stretching sheet, flow through a channel with one wall porous and the other wall permeable, where the solutions were shown to exist only up to a critical value of k, the present problem of flow near a rotating disk admits solution for all values of k, the viscoelastic fluid parameter.

In Figs. 1–4, the profiles of F, G and H are plotted against ζ for $k = 0.01, 0.1, 1$ and 10, respectively. The profiles were also obtained for $k = 100$. However, they are not presented here as the profile for F was almost indistinguishable from the value zero. (We did not give a scaled value of F as it would have given a distorted picture of the profile compared to that for other values of k.) For $k = 0.01$, as expected, the profiles are close to those for Newtonian fluids. But as k is increased, one can note the flattening of the profiles of F and H; also F and G take larger distances from the disk to approach their asymptotic value zero. These conclusions are similar to those for a second-grade fluid, but there is an important difference. For a second-grade fluid the axial velocity at large distances *increases* with the increase in the value of K, the non-Newtonian fluid parameter; for an elastico-viscous fluid the axial velocity at large distances *decreases* with an increase in the value of k. This should not be surprising in view of the fact that the signs of K and k are opposite.

In Fig. 5, the values of $F'(0)$, $G'(0)$ and $H(\infty)$ are plotted against k. It can be seen that they all decrease in size as the value of k is increased. Further for large values of k an asymptotic trend is clearly discernible for each of these parameters. This observation allows us to derive the asymptotic solution for large k, which is presented in Section 5.

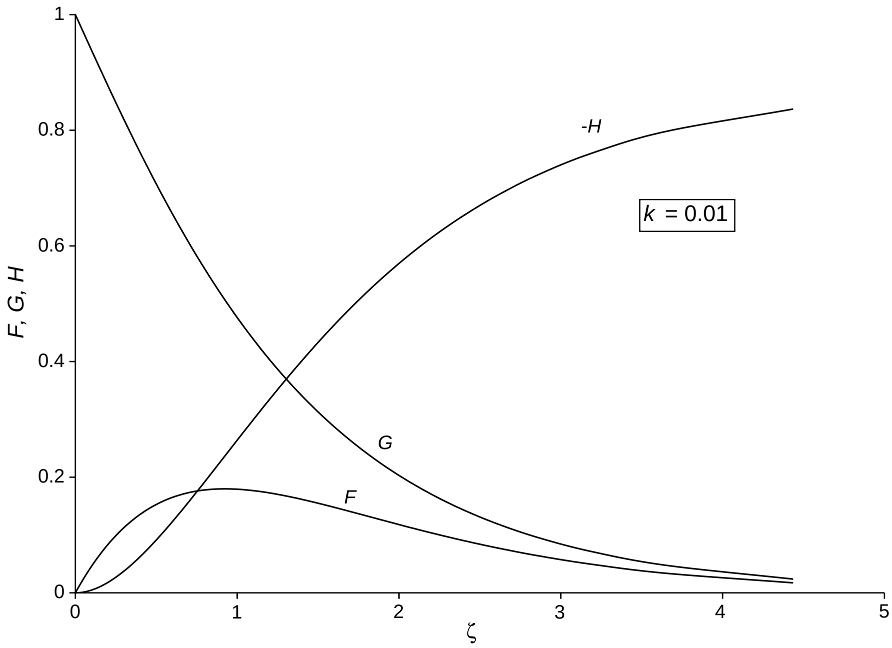

Fig. 1. Variation of F, G and H with ζ, the axial dimensionless distance for $k=0.01$, the viscoelastic fluid parameter.

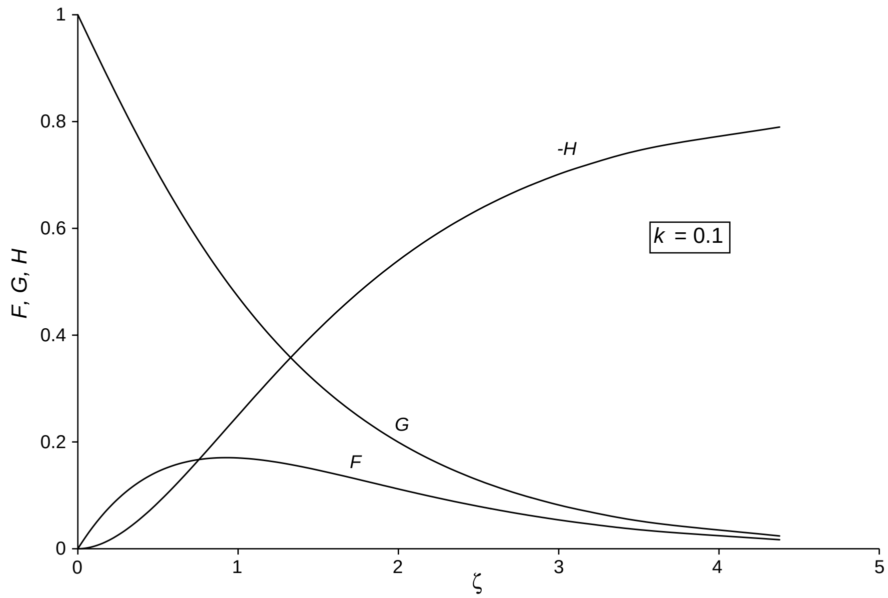

Fig. 2. Variation of F, G and H with ζ, the axial dimensionless distance for $k=0.1$, the viscoelastic fluid parameter.

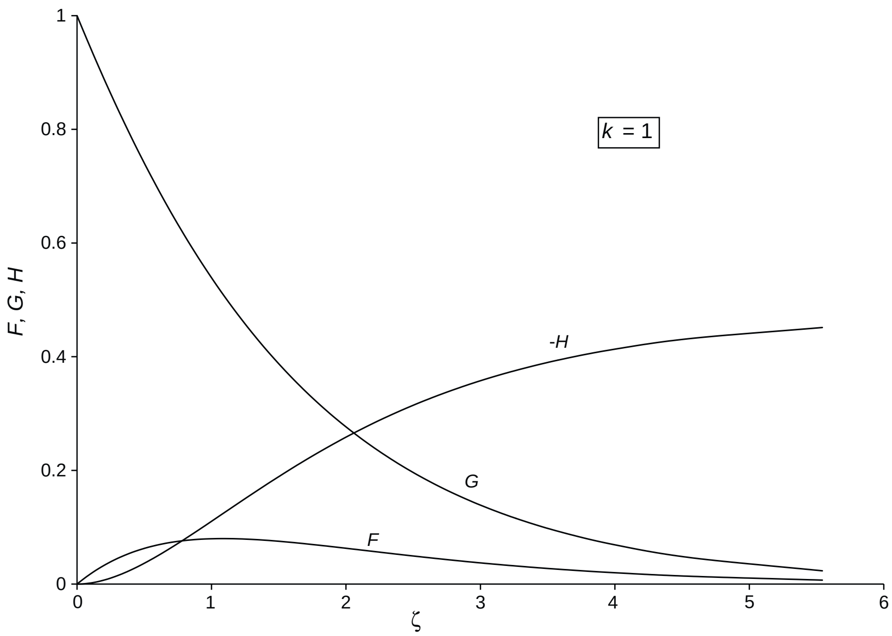

Fig. 3. Variation of F, G and H with ζ, the axial dimensionless distance for $k = 1$, the viscoelastic fluid parameter.

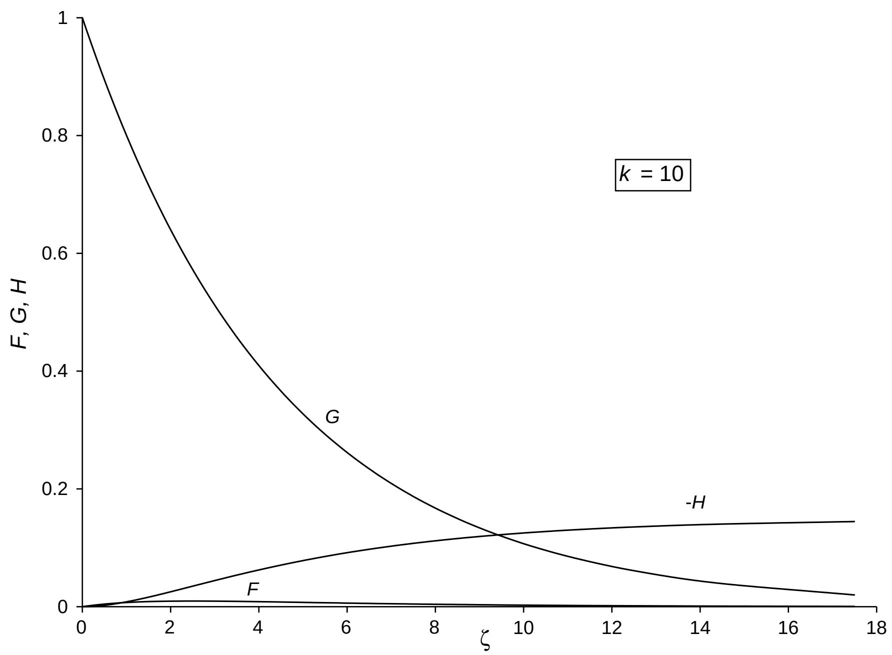

Fig. 4. Variation of F, G and H with ζ, the axial dimensionless distance for $k = 10$, the viscoelastic fluid parameter.

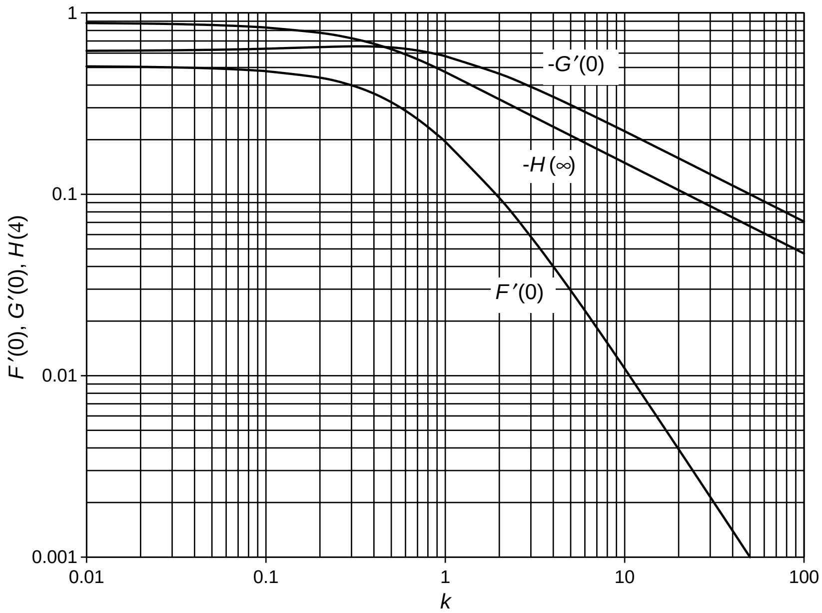

Fig. 5. Variation of $F'(0)$, $G'(0)$ and $H(\infty)$ with k, the viscoelastic fluid parameter.

4. A perturbation solution for small k

It has been pointed out in [2] that the first-order perturbation solution presents rather a misleading picture of the variation of $-G'(0)$, the measure of the turning moment of the disk, with k. Instead of first increasing and, then after attaining a maximum value, decreasing, the first-order perturbation solution, expectantly, gives only the linearly increasing behavior near $k = 0$. It would be of interest to find out if the second- and higher-order perturbation solutions can alter the above picture.

We expand

$$F(\zeta) = \sum_{n=0}^{\infty} F_n(\zeta)k^n, \quad G(\zeta) = \sum_{n=0}^{\infty} G_n(\zeta)k^n, \quad H(\zeta) = \sum_{n=0}^{\infty} H_n(\zeta)k^n \tag{47}$$

and substitute in Eqs. (9)–(13). On equating the like powers of k, we obtain the following perturbation equations:

$$\begin{aligned}
&F_0'' - H_0F_0' - F_0^2 + G_0^2 = 0,\\
&G_0'' - H_0G_0' - 2F_0G_0 = 0,\\
&2F_0 + H_0' = 0,\\
&F_0(0) = 0, \quad G_0(0) = 1, \quad H_0(0) = 0,\\
&F_0(\infty) = 0, \quad G_0(\infty) = 0
\end{aligned} \tag{48}$$

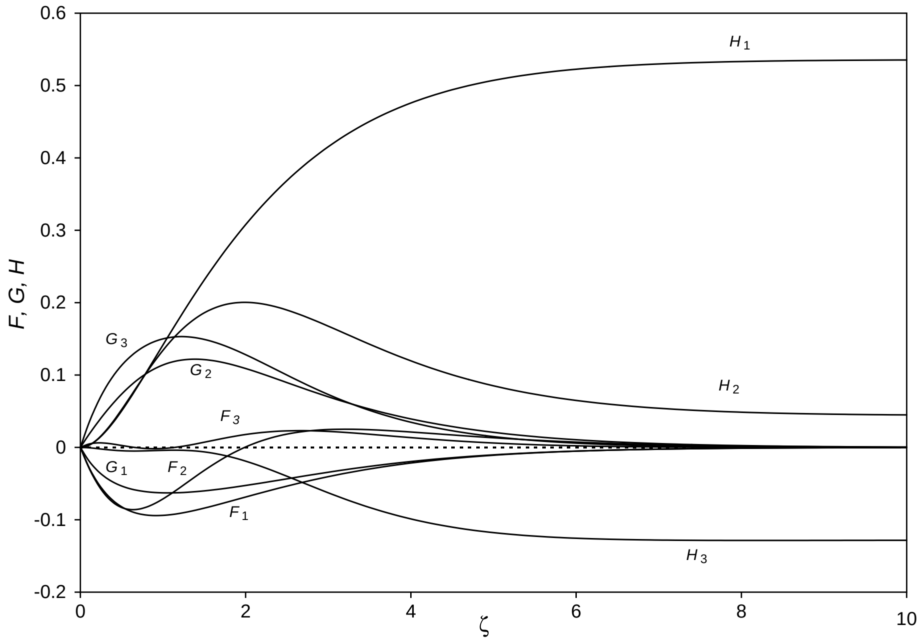

Fig. 6. Variation of the first three order perturbation solutions F_1, G_1 and H_1; F_2, G_2 and H_2 and F_3, G_3 and H_3 in F, G and H (the solution for a Newtonian fluid), respectively, with ζ, the axial dimensionless distance.

and for $n \geqslant 1$

$$F_n'' - \sum_{m=0}^{n}(H_m F_{n-m}' + F_m F_{n-m} - G_m G_{n-m}) - \sum_{m=0}^{n-1}(H_m F_{n-m-1}''' + 4F_m F_{n-m-1}'' + 2G_m G_{n-m-1}'') = 0,$$

$$G_n'' - \sum_{m=0}^{n}(H_m G_{n-m}' + 2F_m G_{n-m}) - \sum_{m=0}^{n-1}(H_m G_{n-m-1}''' + 4F_m G_{n-m-1}'' - 2F_m' G_{n-m-1}') = 0,$$

$$2F_n + H_n' = 0,$$

$$F_n(0) = 0, \quad G_n(0) = 0, \quad H_n(0) = 0,$$

$$F_n(\infty) = 0, \quad G_n(\infty) = 0. \tag{49}$$

Note that BVPs (48) and (49) do not ‘suffer’ from the drawback of having the order of their differential equations greater than the number of boundary conditions. Therefore, they can be integrated easily using any of the standard integration routines such as Runge–Kutta or Adam–Moulton’s method. The solution for the zeroth-order system is well known and one can, for example, refer to [16] for details. The solution for the first-order system has been given by Elliott [10], though due to an error in his analysis his profiles have the opposite sign to those of the actual profiles. We have further carried out the perturbation analysis by obtaining the solutions for the second- and third-order systems. In Fig. 6, the first three perturbations in F, G and H are depicted against ζ. These being

Table 1
Illustrating the variation of $F'(0)$ with k, the viscoelastic fluid parameter using: (i) exact solution, (ii) perturbation solution for small k, and (iii) asymptotic solution for large k

k	$F'(0)$		
	Exact	Perturbation	Asymptotic
0	0.510233	0.510233	
0.1	0.477251	0.477100	
0.2	0.439413	0.438341	
0.3	0.399032	0.394329	
0.4	0.359039	0.345434	
0.5	0.321846	0.292026	
0.6	0.288704	0.234478	
0.7	0.259839	0.173160	
0.8	0.234918	0.108443	
0.9	0.213419	0.040698	
1	0.194811		
2	0.095485		0.189331
5	0.029507		0.029782
10	0.010959		0.010961
20	0.003932		0.003932
50	0.000999		0.000999
100	0.000353		0.000353

universal curves, the values of F, G and H can be obtained for an arbitrary 'small' value of k. To judge how small k can be chosen, the expressions for $F'(0)$, $G'(0)$ and $H(\infty)$ are presented below

$$\begin{aligned} F'(0) &= 0.510233 - 0.301971k - 0.299805k^2 + 0.061840k^3 + \mathrm{O}(k^4), \\ G'(0) &= -0.615922 - 0.208168k + 0.178605k^2 + 0.358044k^3 + \mathrm{O}(k^4), \\ H(\infty) &= -0.884449 + 0.536794k + 0.042214k^2 - 0.127300k^3 + \mathrm{O}(k^4). \end{aligned} \tag{50}$$

These values are also tabulated in Tables 1–3. It can be seen that if we somewhat arbitrarily decide on a tolerance of 2% then the profiles of F, G and H can be accepted for values of k up to 0.35, 0.50 and 0.55, respectively. In Figs. 7–9, $F'(0)$, $G'(0)$ and $H(\infty)$ are shown, respectively, against k for small values of $k(\leqslant 1)$ using various number of terms in expansion (50). It can be seen that whereas the linear perturbation analysis fails to predict the concave down behavior of $-G'(0)$ with k, the second- and higher-order perturbations correctly portray the proper behavior. Thus, in the qualitative sense at any rate the second-order perturbation provides a satisfactory explanation of the behavior of the turning moment of the disk with the viscoelastic fluid parameter.

Table 2
Illustrating the variation of $G'(0)$ with k, the viscoelastic fluid parameter using: (i) exact solution, (ii) perturbation solution for small k, and (iii) asymptotic solution for large k

k	$-G'(0)$		
	Exact	Perturbation	Asymptotic
0	0.615922	0.615922	
0.1	0.634652	0.634596	
0.2	0.647704	0.647547	
0.3	0.653622	0.652631	
0.4	0.652351	0.647698	
0.5	0.645205	0.630599	
0.6	0.634081	0.599187	
0.7	0.620686	0.551314	
0.8	0.606241	0.484831	
0.9	0.591517	0.397589	
1	0.576964	0.287441	1.922447
2	0.461865		0.506836
5	0.310721		0.311026
10	0.222534		0.222538
20	0.157918		0.157918
50	0.099980		0.099980
100	0.070707		0.070707

Table 3
Illustrating the variation of $H(\infty)$ with k, the viscoelastic fluid parameter using: (i) exact solution, (ii) perturbation solution for small k, and (iii) asymptotic solution for large k

k	$-H(\infty)$		
	Exact	Perturbation	Asymptotic
0	0.884449	0.884449	
0.1	0.830507	0.830475	
0.2	0.776760	0.776420	
0.3	0.724605	0.723049	0.860663
0.4	0.675517	0.671124	0.745356
0.5	0.630606	0.621411	0.666667
0.6	0.590374	0.574672	0.608581
0.7	0.554788	0.531672	0.563436
0.8	0.523498	0.493174	0.527046
0.9	0.496019	0.459943	0.496904
1	0.471845	0.432741	0.471405
2	0.333767	0.660405	0.333333
5	0.210823		0.210819
10	0.149071		0.149071
20	0.105409		0.105409
50	0.066667		0.066667
100	0.047140		0.047140

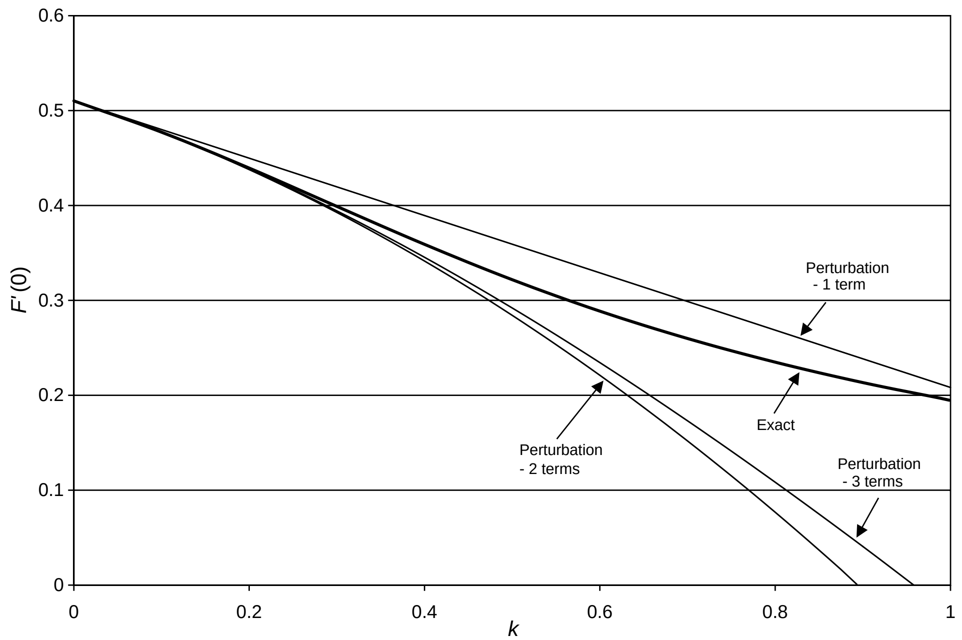

Fig. 7. Variation of $F'(0)$ with k for small values of $k(\leqslant 1)$, using the exact solution and the perturbation solution (50) on taking various number of terms in the expansion.

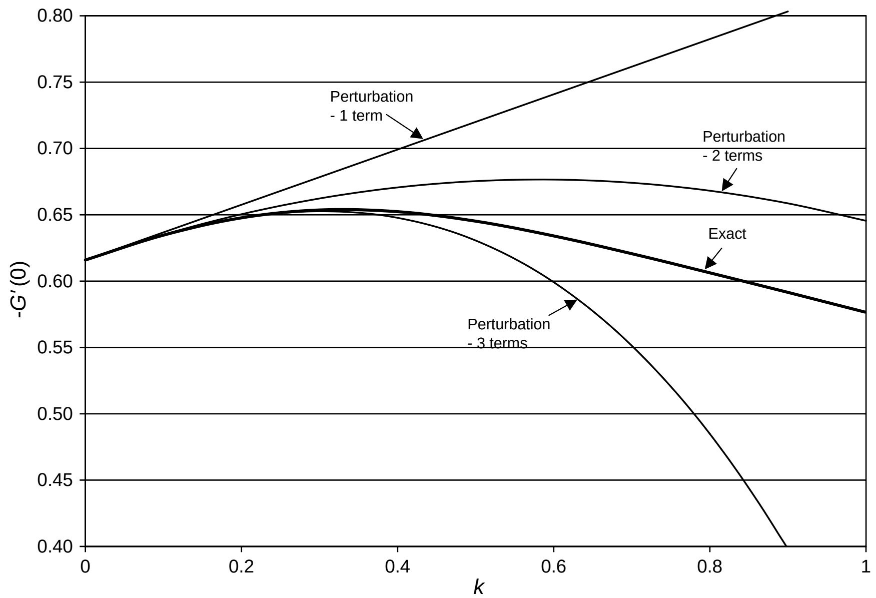

Fig. 8. Variation of $G'(0)$ with k for small values of $k(\leqslant 1)$, using the exact solution and the perturbation solution (50) on taking various number of terms in the expansion.

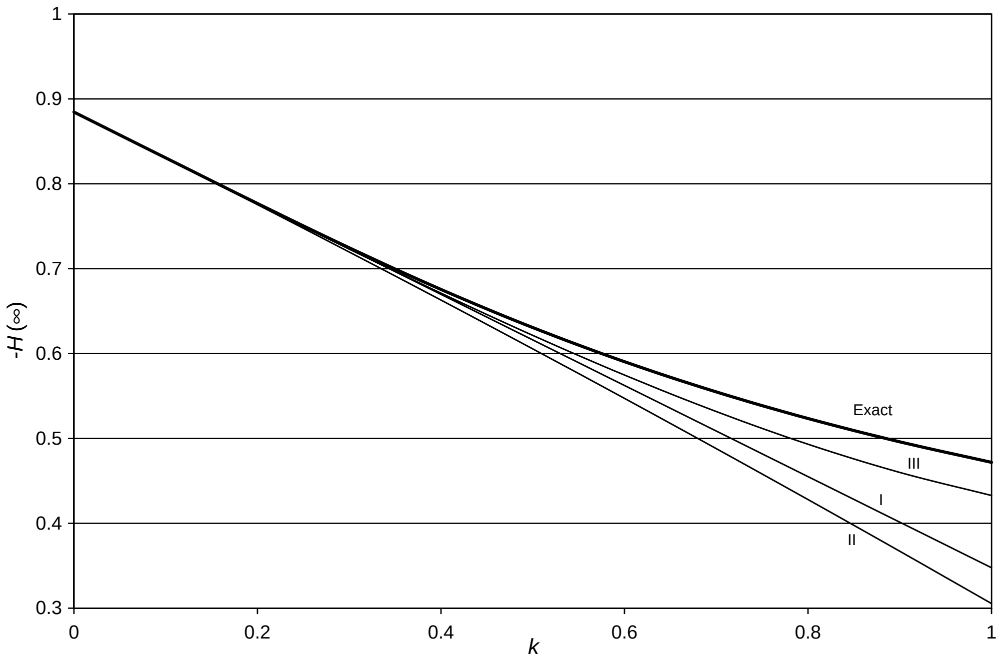

Fig. 9. Variation of $H(\infty)$ with k for small values of $k(\leqslant 1)$, using the exact solution and the perturbation solution (50) upon taking various number of terms in the expansion. The curves labeled I, II, and III denote the solutions on taking one, two and three terms in the expansion, respectively.

5. An asymptotic solution for large *k*

A glance at Fig. 5, in which $F'(0)$, $G'(0)$ and $H(\infty)$ have been plotted against k on a log–log scale, reveals that for large k the graphs of each of them is linear, with $-G'(0)$ and $-H(\infty)$ having a slope of $-\frac{1}{2}$ each, and $F'(0)$ having a slope of $-\frac{3}{2}$. This suggests that

$$F'(0) = \mathrm{O}\left(\frac{1}{k\sqrt{k}}\right), \quad G'(0) = \mathrm{O}\left(\frac{1}{\sqrt{k}}\right) \quad \text{and} \quad H(\infty) = \mathrm{O}\left(\frac{1}{\sqrt{k}}\right) \quad \text{for } k \gg 1. \tag{51}$$

If one goes a step further and estimates the intercept of $-H(\infty)$ (or else uses the interpolation in Table 3), it is easy to verify that

$$H(\infty) = -\frac{1}{3}\sqrt{\frac{2}{k}} \quad \text{for } k \gg 1. \tag{52}$$

We shall be verifying the validity of statement (52) shortly.

Eq. (52) allows us to estimate c from Eq. (21). We have

$$c = \frac{1}{\sqrt{2k}} \quad \text{for } k \gg 1. \tag{53}$$

Therefore, we define

$$\eta = \frac{\zeta}{\sqrt{2k}}, \tag{54}$$

which transforms Eqs. (9)–(11) into

$$\frac{1}{2k}\frac{d^2F}{d\eta^2} - \frac{H}{\sqrt{2k}}\frac{dF}{d\eta} - F^2 + G^2 - \frac{H}{2\sqrt{2k}}\frac{d^3F}{d\eta^3} - 2F\frac{d^2F}{d\eta^2} - \left(\frac{dG}{d\eta}\right)^2 = 0, \tag{55}$$

$$\frac{1}{2k}\frac{d^2G}{d\eta^2} - \frac{H}{\sqrt{2k}}\frac{dG}{d\eta} - 2FG - \frac{H}{2\sqrt{2k}}\frac{d^3G}{d\eta^3} - 2F\frac{d^2G}{d\eta^2} + \frac{dF}{d\eta}\frac{dG}{d\eta} = 0, \tag{56}$$

$$2F + \frac{1}{\sqrt{2k}}\frac{dH}{d\eta} = 0. \tag{57}$$

We shall further find it convenient to introduce

$$W = \frac{H}{2\sqrt{2k}}, \tag{58}$$

that allows Eqs. (55)–(57) to be rewritten as

$$\frac{1}{2k}\frac{d^2F}{d\eta^2} - 2W\frac{dF}{d\eta} - F^2 + G^2 - W\frac{d^3F}{d\eta^3} - 2F\frac{d^2F}{d\eta^2} - \left(\frac{dG}{d\eta}\right)^2 = 0, \tag{59}$$

$$\frac{1}{2k}\frac{d^2G}{d\eta^2} - 2W\frac{dG}{d\eta} - 2FG - W\frac{d^3G}{d\eta^3} - 2F\frac{d^2G}{d\eta^2} + \frac{dF}{d\eta}\frac{dG}{d\eta} = 0, \tag{60}$$

$$F + \frac{dW}{d\eta} = 0. \tag{61}$$

The boundary conditions on F, G and W become

$$\text{at} \quad \eta = 0, \quad F = 0, \quad G = 1, \quad W = 0,$$

$$\text{at} \quad \eta \to \infty, \quad F = 0, \quad G = 0. \tag{62}$$

Also in terms of η, the asymptotic behavior of F, G and W can be deduced from Eq. (51) as

$$\frac{dF(0)}{d\eta} = O\left(\frac{1}{k}\right), \quad \frac{dG(0)}{d\eta} = O(1) \quad \text{and} \quad W(\infty) = O\left(\frac{1}{k}\right) \quad \text{for } k \geqslant 1. \tag{63}$$

Accordingly, we assume

$$F = \frac{F^*}{k}, \quad W = \frac{W^*}{k}, \tag{64}$$

where F^* and W^* are O(1).

If we substitute for F and W from (64) into (59)–(61), we obtain

$$G^2 - \left(\frac{dG}{d\eta}\right)^2 + \frac{1}{k^2}\left[\frac{1}{2}\frac{d^2F^*}{d\eta^2} - 2W^*\frac{dF^*}{d\eta} - F^{*2} - W^*\frac{d^3F^*}{d\eta^3} - 2F^*\frac{d^2F^*}{d\eta^2}\right] = 0, \tag{65}$$

$$\frac{1}{2}\frac{d^2G}{d\eta^2} - 2W^*\frac{dG}{d\eta} - 2F^*G - W^*\frac{d^3G}{d\eta^3} - 2F^*\frac{d^2G}{d\eta^2} + \frac{dF^*}{d\eta}\frac{dG}{d\eta} = 0, \tag{66}$$

$$F^* + \frac{dW^*}{d\eta} = 0 \tag{67}$$

and the boundary conditions (62) take the form

$$\begin{aligned}
&\text{at} \quad \eta = 0: \quad F^* = 0, \quad G = 1, \quad W^* = 0,\\
&\text{at} \quad \eta \to \infty: \quad F^* = 0, \quad G = 0.
\end{aligned} \tag{68}$$

Eqs. (65) and (66) suggest that we take

$$\begin{aligned}
F^* &= F_0 + \frac{F_1}{k^2} + \frac{F_2}{k^4} + \cdots,\\
G &= G_0 + \frac{G_1}{k^2} + \frac{G_2}{k^4} + \cdots,\\
W^* &= W_0 + \frac{W_1}{k^2} + \frac{W_2}{k^4} + \cdots \ .
\end{aligned} \tag{69}$$

(The subscripted F and G used in (69) should not be confused with those used in the preceding section for small values of k). Substituting for F^*, G and W^* from (69) into (65)–(68) yields the following system of BVPs:

$$G_0^2 - \left(\frac{\mathrm{d}G_0}{\mathrm{d}\eta}\right)^2 = 0, \tag{70}$$

$$\begin{aligned}
&\sum_{m=0}^{n}\left(G_m G_{n-m} - \frac{\mathrm{d}G_m}{\mathrm{d}\eta}\frac{\mathrm{d}G_{n-m}}{\mathrm{d}\eta}\right) + \frac{1}{2}\frac{\mathrm{d}^2 F_{n-1}}{\mathrm{d}\eta^2}\\
&\quad - \sum_{m=0}^{n-1}\left(2W_m\frac{\mathrm{d}F_{n-m-1}}{\mathrm{d}\eta} - F_m F_{n-m-1} - W_m\frac{\mathrm{d}^3 F_{n-m-1}}{\mathrm{d}\eta^3} - 2F_m\frac{\mathrm{d}^2 F_{n-m-1}}{\mathrm{d}\eta^2}\right) = 0, \quad n \geqslant 1
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}^2 G_n}{\mathrm{d}\eta^2} - \sum_{m=0}^{n}\left(2W_m\frac{\mathrm{d}G_{n-m}}{\mathrm{d}\eta} + 2F_m G_{n-m} + W_m\frac{\mathrm{d}^3 G_{n-m}}{\mathrm{d}\eta^3} + 2F_m\frac{\mathrm{d}^2 G_{n-m}}{\mathrm{d}\eta^2} - \frac{\mathrm{d}F_m}{\mathrm{d}\eta}\frac{\mathrm{d}G_n}{\mathrm{d}\eta}\right) = 0,\\
&\quad n \geqslant 0,
\end{aligned} \tag{72}$$

$$F_n + \frac{\mathrm{d}W_n}{\mathrm{d}\eta} = 0, \quad n \geqslant 0, \tag{73}$$

$$\begin{aligned}
&F_0(0) = 0, \quad G_0(0) = 1, \quad W_0(0) = 0,\\
&F_0(\infty) = 0, \quad G_0(\infty) = 0,
\end{aligned} \tag{74}$$

$$\begin{aligned}
&F_n(0) = 0, \quad G_n(0) = 0, \quad W_n(0) = 0,\\
&F_n(\infty) = 0, \quad G_n(\infty) = 0
\end{aligned} \tag{75}$$

for $n \geqslant 1$.

The only solution of Eq. (70) satisfying boundary condition (73) is

$$G_0 = \mathrm{e}^{-\eta},$$

which when substituted in Eqs. (71)–(73) results into the following linear system in G_n and F_n:

$$\frac{\mathrm{d}G_n}{\mathrm{d}\eta} + G_n = -\frac{1}{2}\,\mathrm{e}^{\eta}\left[\sum_{m=1}^{n-1}\left(G_m G_{n-m} - \frac{\mathrm{d}G_m}{\mathrm{d}\eta}\,\frac{\mathrm{d}G_{n-m}}{\mathrm{d}\eta}\right) + \frac{1}{2}\,\frac{\mathrm{d}^2F_{n-1}}{\mathrm{d}\eta^2}\right.$$
$$\left. - \sum_{m=0}^{n-1}\left(2W_m\,\frac{\mathrm{d}F_{n-m-1}}{\mathrm{d}\eta} - F_mF_{n-m-1} - W_m\,\frac{\mathrm{d}^3F_{n-m-1}}{\mathrm{d}\eta^3} - 2F_m\,\frac{\mathrm{d}^2F_{n-m-1}}{\mathrm{d}\eta^2}\right)\right], \quad n \geqslant 1, \tag{76}$$

$$\frac{\mathrm{d}F_n}{\mathrm{d}\eta} + 4F_n - 3W_n = \mathrm{e}^{\eta}\left[\frac{1}{2}\,\frac{\mathrm{d}^2G_n}{\mathrm{d}\eta^2} - \sum_{m=1}^{n-1}\left(2W_m\,\frac{\mathrm{d}G_{n-m}}{\mathrm{d}\eta} + 2F_mG_{n-m}\right.\right.$$
$$\left.\left. + W_m\,\frac{\mathrm{d}^3G_{n-m}}{\mathrm{d}\eta^3} + 2F_m\,\frac{\mathrm{d}^2G_{n-m}}{\mathrm{d}\eta^2} - \frac{\mathrm{d}F_m}{\mathrm{d}\eta}\,\frac{\mathrm{d}G_n}{\mathrm{d}\eta}\right)\right], \quad n \geqslant 0. \tag{77}$$

It is interesting to note that Eq. (9) which is the 'differential equation' for F yields Eq. (76) that will be used to solve for G_n. Conversely, Eq. (10) which is the 'differential equation' for G yields Eq. (77) that will be used to solve for F_n. The procedure for obtaining the asymptotic solutions is straightforward. First one solves for F_0 from Eqs. (77) and (73) making use of boundary conditions (74). Then one alternates between Eqs. (76) and (77) to obtain the values of $G_1, F_1, G_2, F_2, \ldots$, etc. The solutions are listed below

$$G_0 = \mathrm{e}^{-\eta}, \quad W_0 = \frac{1}{12}(-2 + 3\mathrm{e}^{-\eta} - \mathrm{e}^{-3\eta}), \quad F_0 = \frac{1}{4}(\mathrm{e}^{-\eta} - \mathrm{e}^{-3\eta}), \tag{78}$$

$$\begin{aligned}
G_1 &= \frac{1}{8}\,\mathrm{e}^{-\eta} + \frac{1}{8}\,\mathrm{e}^{-2\eta} - \frac{3}{16}\,\mathrm{e}^{-3\eta} - \frac{1}{16}\,\mathrm{e}^{-5\eta},\\
W_1 &= \frac{17}{192}\,\mathrm{e}^{-\eta} + \frac{5}{16}\,\mathrm{e}^{-2\eta} - \left(\frac{23}{64} + \frac{3}{8}\,\eta\right)\mathrm{e}^{-3\eta} - \frac{5}{24}\,\mathrm{e}^{-4\eta} + \frac{11}{64}\,\mathrm{e}^{-5\eta} - \frac{1}{192}\,\mathrm{e}^{-7\eta},\\
F_1 &= \frac{17}{192}\,\mathrm{e}^{-\eta} + \frac{5}{8}\,\mathrm{e}^{-2\eta} - \left(\frac{45}{64} + \frac{9}{8}\,\eta\right)\mathrm{e}^{-3\eta} - \frac{5}{6}\,\mathrm{e}^{-4\eta} + \frac{55}{64}\,\mathrm{e}^{-5\eta} - \frac{7}{192}\,\mathrm{e}^{-7\eta},
\end{aligned} \tag{79}$$

$$\begin{aligned}
G_2 = {}& -\frac{221}{960}\,\mathrm{e}^{-\eta} + \left(-\frac{1}{64} + \frac{9}{16}\,\eta\right)\mathrm{e}^{-2\eta} + \left(\frac{769}{768} - \frac{27}{32}\,\eta\right)\mathrm{e}^{-3\eta} - \frac{475}{192}\,\mathrm{e}^{-4\eta}\\
& + \left(\frac{241}{128} - \frac{9}{16}\,\eta\right)\mathrm{e}^{-5\eta} - \frac{259}{960}\,\mathrm{e}^{-6\eta} + \frac{83}{768}\,\mathrm{e}^{-7\eta} - \frac{1}{384}\,\mathrm{e}^{-9\eta},
\end{aligned}$$

$$
\begin{aligned}
W_2 = & -\left(\frac{1641}{7168}+\frac{3}{32}\,\eta\right) e^{-\eta}-\left(\frac{907}{256}-\frac{45}{32}\,\eta\right) e^{-2\eta}-\left(\frac{20533}{3840}-\frac{3041}{384}\,\eta+\frac{27}{32}\,\eta^2\right) e^{-3\eta} \\
& +\left(\frac{3587}{256}-\frac{15}{8}\,\eta\right) e^{-4\eta}-\left(\frac{5795}{1024}-\frac{99}{64}\,\eta\right) e^{-5\eta}+\frac{539}{576}\,e^{-6\eta}-\left(\frac{155}{1024}+\frac{9}{128}\,\eta\right) e^{-7\eta} \\
& -\frac{209}{4480}\,e^{-8\eta}+\frac{137}{4608}\,e^{-9\eta}-\frac{1}{1024}\,e^{-11\eta}, \qquad (80) \\
F_2 = & -\left(\frac{969}{7168}+\frac{3}{32}\,\eta\right) e^{-\eta}-\left(\frac{1089}{128}-\frac{45}{16}\,\eta\right) e^{-2\eta}-\left(\frac{92009}{3840}-\frac{3257}{128}\,\eta+\frac{81}{32}\,\eta^2\right) e^{-3\eta} \\
& +\left(\frac{3707}{64}-\frac{15}{2}\,\eta\right) e^{-4\eta}-\left(\frac{30559}{1024}-\frac{495}{64}\,\eta\right) e^{-5\eta}+\frac{539}{96}\,e^{-6\eta}-\left(\frac{1013}{1024}+\frac{63}{128}\,\eta\right) e^{-7\eta} \\
& -\frac{209}{560}\,e^{-8\eta}+\frac{137}{512}\,e^{-9\eta}-\frac{11}{1024}\,e^{-11\eta} \\
& \vdots \\
& .
\end{aligned}
$$

The higher-order solutions can obviously be obtained but their expressions become increasingly tedious. We also refrain from giving the plots of the universal functions defined by Eqs. (78)–(80) as they can easily be generated by substituting the desired value of η. The asymptotic expansions of the missing initial conditions are

$$
\begin{aligned}
F'(0) &= \frac{1}{\sqrt{2k}}\left[\frac{1}{2k}-\frac{17}{16k^3}+\frac{1071}{128k^5}\right]+\mathrm{O}\left(\frac{1}{k^{15/2}}\right), \\
G'(0) &= -\frac{1}{\sqrt{2k}}\left[1-\frac{1}{2k^2}+\frac{71}{32k^4}\right]+\mathrm{O}\left(\frac{1}{k^{13/2}}\right), \qquad (81) \\
H(\infty) &= -\frac{1}{3}\sqrt{\frac{2}{k}},
\end{aligned}
$$

for $k \gg 1$.

In Fig. 10, $F'(0)$ and $G'(0)$ are drawn against k for large values of $k(>1)$. Besides the exact solution, the asymptotic solutions from Eq. (81), on truncating the expansions after one, two, and three terms, are shown. It can be seen that $G'(0)$, the more important of the two parameters (it represents the turning moment on the disk), has a more accurate asymptotic representation. For the 2% tolerance criterion, the three-term expansion can be considered adequate for values of k as low as 2.80. On the other hand, the corresponding expansion for $F'(0)$ gives comparable accuracy for values of k greater than 4.35. These results can be further corroborated from Tables 1 and 2, where the values of $F'(0)$ and $G'(0)$ obtained from expansion (81) are also presented. It is particularly interesting to note that the value of $H(\infty)$ has only one-term expansion for large k. Our numerical results show that this expansion gives a satisfactory estimate of $H(\infty)$ for all values of $k > 1$.

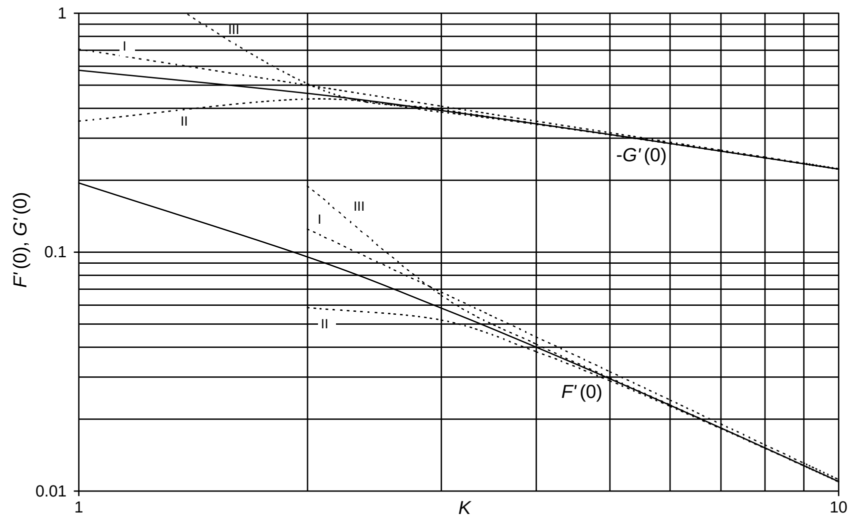

Fig. 10. Variation of $F'(0)$ and $G'(0)$ with k for large values of $k(\geqslant 1)$, using the exact solution and the asymptotic solution (81) upon taking various number of terms in the expansion. The curves labeled I, II, and III denote the solutions on taking one, two and three terms in the expansions, respectively.

We now turn our attention to the numerical solution of BVP (9)–(13) for large values of k. The basis of obtaining the solution is the equivalent system (65)–(68). We set up the iterative scheme

$$G_{\sigma+1}^2 - \left(\frac{\mathrm{d}G_{\sigma+1}}{\mathrm{d}\eta}\right)^2 = \frac{1}{k^2}\left[-\frac{1}{2}\frac{\mathrm{d}^2F_\sigma^*}{\mathrm{d}\eta^2} + 2W^*\frac{\mathrm{d}F_\sigma^*}{\mathrm{d}\eta} + F_\sigma^{*2} + W_\sigma^*\frac{\mathrm{d}^3F_\sigma^*}{\mathrm{d}\eta^3} + 2F_\sigma^*\frac{\mathrm{d}^2F_\sigma^*}{\mathrm{d}\eta^2}\right], \tag{82}$$

$$\frac{\mathrm{d}G_{\sigma+1}}{\mathrm{d}\eta}\frac{\mathrm{d}F_{\sigma+1}^*}{\mathrm{d}\eta} - 2\left(\frac{\mathrm{d}^2G_{\sigma+1}}{\mathrm{d}\eta^2} + G_{\sigma+1}\right)F_{\sigma+1}^* - \left(\frac{\mathrm{d}^3G_{\sigma+1}}{\mathrm{d}\eta^3} + 2\frac{\mathrm{d}G_{\sigma+1}}{\mathrm{d}\eta}\right)W_{\sigma+1}^* = -\frac{1}{2}\frac{\mathrm{d}^2G_{\sigma+1}}{\mathrm{d}\eta^2}, \tag{83}$$

$$F_{\sigma+1}^* + \frac{\mathrm{d}W_{\sigma+1}^*}{\mathrm{d}\eta} = 0, \tag{84}$$

$$\begin{aligned} &F_{\sigma+1}^*(0) = 0, \quad G_{\sigma+1}(0) = 1, \quad W_{\sigma+1}^*(0) = 0, \\ &F_{\sigma+1}^*(\infty) = 0, \quad G_{\sigma+1}(\infty) = 0, \end{aligned} \tag{85}$$

where a subscript denotes the iteration number.

In Eq. (82), $\mathrm{d}G_{\sigma+1}/\mathrm{d}\eta$ is solved explicitly, taking the negative sign of the square root, since G decreases with η. The switching of the equations for F and G results into an unusual feature, namely, the order of the system is reduced to three. Hence, system (82)–(84) is effectively an initial value problem, and therefore can be integrated in a straightforward manner to any desired degree of accuracy.

The iterations are started by taking $F_0^*(\eta) \equiv 0$, and solving Eq. (82) for $G_1(\eta)$, using the initial condition on G_1 in (85). Next, the value of $G_1(\eta)$ is substituted in Eq. (83) and the pair of linear differential equations (83) and (84) are solved for $F_1^*(\eta)$ and $W_1^*(\eta)$ using the appropriate initial conditions in (85). The cycle is repeated till the values of F^*, G and W^* agree to the desired degree of accuracy for each η on two successive iterations. It was found that the iterative scheme converged for all values of k greater than 20. As expected, the convergence got better as the value of k was increased. Whereas for an eight significant digits accuracy it required 20 iterations for $k=20$, the number came down to only 4 iterations for $k=50$. The important implication of the present algorithm is that in conjunction with the one given in Section 3, it allows the numerical solution to be obtained for the entire range of values of k.

6. Conclusions

In the present paper we have considered the laminar flow of an elastico-viscous fluid modeled by Walter B$'$ fluid near a rotating disk. The flow is characterized by a boundary-value problems (BVP) in which the number of available boundary conditions is less than the order of the system of differential equations. Nevertheless, it is possible to solve the BVP numerically without placing any restriction on the size of k. The algorithm given in the paper also successfully negotiates the problem of errors induced on account of replacing numerical infinity by a finite value by reducing the infinite range of integration to $[0,1]$. Moreover, the algorithm is equally applicable for the three cases: (i) $k=0$, (ii) $k\to 0$ and (iii) arbitrary k, except when k becomes very large (>20). In the latter case, it is preferable to use another algorithm which is developed and modeled on the asymptotic solution of the BVP for large values of k.

Unlike most other problems of viscoelastic fluid, such as stagnation point flows, flow through a porous channel, flow parallel to a porous wall, for which solutions are possible only up to some critical value of k, for the problem under consideration, it is possible to find the solution for *all* values of k. One of the important characteristics of the flow, which was also reported earlier [2], is that there is a greater torque on the disk on account of the presence of viscoelasticity of the fluid. This trend is however reversed as the value of k is increased. A linear perturbation analysis fails to explain this behavior. Therefore, in the present paper higher-order perturbation solutions have been derived. The second- and higher-order perturbations satisfactorily demonstrate the above-mentioned behavior, though their utility is restricted for values of k less than unity on account of the loss of accuracy for larger values of k.

The numerical solution exhibits an asymptotic behavior for large k. This allows analytical solutions to be derived for large values of k. Three term asymptotic solutions have been obtained for each component of velocity. It is shown that as k is increased, the radial velocity decays inversely as k, the axial velocity inversely as $\frac{3}{2}$ power of k, and the torque on the disk as the inverse square root of k.

For both the series solutions (perturbation for small k, and asymptotic for large k) the appropriate ranges of k for which the expansions are valid are singled out.

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